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# Space-like coherent states of time-dependent Morse oscillator

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**Abstract.** On the basis of the Feinberg-Horodecki quantal equation the space-like coherent states of a time-dependent Morse oscillator, minimizing the time-energy uncertainty relation are constructed. They reduce to the macroscopic states of the Gompertzian growth, in the limit of the anharmonicity constant $x_e \rightarrow 1$. The obtained results are useful for interpreting the formation of the specific growth patterns during crystallization process and biological growth.

**PACS.** 03.65.Fd Algebraic methods – 05.60.-k Transport processes – 81.10.Aj Theory and models of crystal growth; physics of crystal growth, crystal morphology, and orientation – 89.75.-k Complex systems

## 1 Introduction

In the past two and a half decades, much effort has been undertaken to construct coherent states for general anharmonic potentials, particularly the Morse potential [1–9]. Such coherent states are usually constructed using a transformation of the basic Hamiltonian to the form resembling that for an harmonic oscillator [2], or by an algebraic method [8] employing the supersymmetric quantum mechanics [7,9]. A point of departure for the above techniques is the time-like Schrödinger equation including space-dependent potentials. In this work we extend the research area onto coherent states of the time-dependent Morse oscillator, constructed on the basis of the space-like counterpart of the Schrödinger equation [10].

The time-like coherent states of the space-dependent oscillators are defined as: (i) eigenstates of the annihilation operator, (ii) states that minimize the position-momentum uncertainty relation and (iii) states that arise from the operation of a unitary displacement operator to the ground state of the oscillator [11]. The space-like coherent states differ from the time-like ones as they minimize the time-energy uncertainty relation. Such states have not been considered yet, as they are difficult to interpret in terms of temporal bound states and vibrational motion. However, in the case of anharmonic oscillators, we have no bound states in the dissociation limit and the direction of temporal motion is consistent with the arrow of time (is not of the oscillatory type). In view of this, the main objective of the present work is to construct the space-like coherent states for the time-dependent Morse oscillator employing the Feinberg-Horodecki equation being the space-like counterpart of the time-like Schrödinger equation. It will be proved that in the dissociation limit the space-like Morse coherent states reduce to the well-known macroscopic states of the Gompertzian growth whose temporal evolution is described by the Gompertz [12] or Zwietering et al. [13] functions. The results obtained are useful in explanation of the origin of the Gompertzian growth and formation of the specific growth patterns in biological and inorganic systems, particularly the growth of organisms, tumors, bacterial colonies and crystals.

## 2 Time- and space-like fields

The time- and space-like coherent states and equations governing their propagation are symmetric with respect to time and space coordinates, hence a consistent introduction of the latter is possible within the generalized quantum theory including the space-like quantum states. A relativistic version of such a theory has been introduced by Feinberg [14], whereas its nonrelativistic approximation has been obtained by Horodecki [10], who derived the space-like counterpart of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial x}\Psi = -\frac{\hbar^2}{2m_0c}\frac{\partial^2}{\partial x_0^2}\Psi + \frac{V}{c}\Psi \qquad (1)$$

called the Feinberg-Horodecki equation. Here, $V$ denotes the vector potential, $m_0$ is the mass of the particle, $x_0 = ct$ whereas $c$ is the light velocity.

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In the case of stationary states with the momentum $P = const$, the space-independent version of the Feinberg-Horodecki equation (1) is derivable by substitution [10] of

$$\Psi = \phi(t)e^{-iPx/\hbar} \tag{2}$$

into (1) yielding

$$-\frac{\hbar^2}{2m_0c^3}\frac{\partial^2}{\partial t^2}\phi(t) + \frac{V}{c}\phi(t) = P\phi(t). \tag{3}$$

This is identical to an energy eigenvalue equation in a scalar potential $V$, with $x$ replaced by $t$.

The Feinberg-Horodecki equation and space-like systems deserve attention as they play an important role in the extended special relativity and extended quantum mechanics [15–17]. In particular, they are useful in explanation of the conversion of light in ponderable matter [18], the nature of electric charge and the force binding charge and mass in a stable particle [19] and other phenomena difficult to explain in the framework of orthodoxal time-like physics [15,16].

## 3 The space-like coherent states of Morse oscillator

The coherent states of a space-like anharmonic Morse oscillator can be constructed for the temporal counterpart of the spatial Morse potential employing the algebraic procedure proposed by Cooper [8]. The point of departure is the Feinberg-Horodecki quantal equation

$$-\frac{\hbar^2}{2m_0c}\frac{\partial^2}{\partial x_0^2}\phi(x_0) + V(x_0)\phi(x_0)/c = P\phi(x_0), \tag{4}$$

including the time-dependent Morse vector potential

$$V(x_0) = D_0\left(1 - e^{-\kappa x_0}\right)^2 = D_0\left(1 - e^{-at}\right)^2 \tag{5}$$

in which $D_0$ and $a = c\kappa$ denote the dissociation energy and the range parameter, respectively.

Applying the Morse procedure [20] to (4), one gets the momentum eigenvalues of the Feinberg-Horodecki-Morse equation

$$P = \hbar K[(v + 1/2) - (v + 1/2)^2 x_e] \quad v = 0, 1, 2... \tag{6}$$

in which

$$K = \frac{\omega}{c} = \frac{\kappa}{c}\sqrt{\frac{2D_0}{m_0}}, \quad x_e = \frac{\hbar\omega}{4D_0} \tag{7}$$

are the wavevector and the anharmonicity constant, respectively, whereas $\omega$ is the vibrational frequency. Taking advantage of the dimensionless coordinate

$$\tau = \sqrt{\frac{m_0\omega}{\hbar}}x_0 \tag{8}$$

we arrive at the quantal equation

$$-\frac{1}{2}\frac{d^2\phi_v}{d\tau^2} + \frac{1}{4x_e}\left(1 - e^{-\sqrt{2x_e}\tau}\right)^2\phi_v = [(v + 1/2) - (v + 1/2)^2 x_e]\phi_v, \tag{9}$$

which can be expressed in the factorized form [8]

$$\hat{B}^\dagger\hat{B}|v\rangle = v[1 - (v + 1)x_e]|v\rangle. \tag{10}$$

Here

$$\hat{B} = \frac{1}{\sqrt{2}}\left[\frac{d}{d\tau} + \frac{1}{\sqrt{2x_e}}\left(1 - e^{-\sqrt{2x_e}\tau}\right) - \sqrt{\frac{x_e}{2}}\right] \tag{11}$$

$$\hat{B}^\dagger = \frac{1}{\sqrt{2}}\left[-\frac{d}{d\tau} + \frac{1}{\sqrt{2x_e}}\left(1 - e^{-\sqrt{2x_e}\tau}\right) - \sqrt{\frac{x_e}{2}}\right] \tag{12}$$

are the space-like annihilation and creation operators, respectively.

The space-like coherent states of the Morse oscillator are eigenstates of the annihilation operator

$$\hat{B}|\beta\rangle = \beta|\beta\rangle \tag{13}$$

$$|\beta\rangle = e^{\sqrt{2}\beta\tau}\exp\left[-\frac{1}{2x_e}e^{-\sqrt{2x_e}\tau}\right]e^{-\frac{1}{\sqrt{2x_e}}(1-x_e)\tau}. \tag{14}$$

Such states minimize the time-energy uncertainty relation

$$(\Delta\tau)^2(\Delta E)^2 \geq \frac{1}{4}\langle\beta|g(\tau)|\beta\rangle^2,$$
$$[\tau, \hat{E}] = ig(\tau), \quad g(\tau) = e^{-\frac{1}{\sqrt{2x_e}}\tau}, \tag{15}$$

yielding

$$(\Delta\tau)^2(\Delta E)^2 = \frac{1}{4}\langle\beta|g(\tau)|\beta\rangle^2 \tag{16}$$

in which

$$(\Delta\tau)^2 = \langle\beta|\tau^2|\beta\rangle - \langle\beta|\tau|\beta\rangle^2,$$
$$(\Delta E)^2 = \langle\beta|\hat{E}^2|\beta\rangle - \langle\beta|\hat{E}|\beta\rangle^2, \tag{17}$$

$$\hat{E} = \frac{d}{d\tau} \quad (\hbar = 1), \quad \tau = \frac{1}{\sqrt{2x_e}}e^{-\sqrt{2x_e}\tau} + \frac{1}{\sqrt{2x_e}}(1 - x_e), \tag{18}$$

whereas $|\beta\rangle$ is given by (14). In the above formulae, $\tau$ denotes, to within a constant, the temporal dimensionless Morse variable.

## 4 The dissociation limit $x_e \to 1$

In the theory of time-like coherent states of the Morse oscillator the anharmonicity parameter $x_e$ plays an important role as for $x_e \to 0$ the harmonic limit is regained [8]. On the other hand, for $x_e \to 1$ one can get from (9) the dissociation limit equation

$$-\frac{1}{2}\frac{d^2\phi_v}{d\tau^2} + \frac{1}{4}\left(1 - e^{-\sqrt{2}\tau}\right)^2\phi_v = \left(\frac{1}{4} - v^2\right)\phi_v \tag{19}$$

for the space-like Morse oscillator with the anharmonicity constant $x_e = 1$. The eigenvalues of the above equation are related to the quantized momentum

$$P_v = \hbar K \left(\frac{1}{4} - v^2\right), \tag{20}$$

satisfying the relationship

$$\Delta P_{v0} = P_v - P_0 = -\hbar K v^2. \tag{21}$$

Since $\Delta P_{v0}$ cannot be negative, equation (19) specified in the form

$$-\frac{1}{2}\frac{d^2\phi_0}{d\tau^2} + \frac{1}{4}\left(1 - e^{-\sqrt{2}\tau}\right)^2 \phi_0 = \frac{1}{4}\phi_0 \tag{22}$$

has only one ground-state solution for $v = 0$

$$\phi_0 = e^{-\frac{1}{2}e^{-\sqrt{2}\tau}}, \tag{23}$$

associated with the nonvanishing momentum $P = \hbar K/4$. Returning to original time-coordinate and taking advantage of the relation

$$x_e = \frac{\hbar\omega}{4D_0} = 1 \tag{24}$$

equation (22) can be transformed to the form

$$-\frac{\hbar^2}{2m_0c^2}\frac{d^2\phi_0}{dt^2} + D_0\left(1 - e^{-at}\right)^2 \phi_0 = D_0\phi_0 \tag{25}$$

or alternatively

$$-\frac{\hbar^2}{2m_0c^3}\frac{d^2\phi_0}{dt^2} + P_0\left(1 - e^{-at}\right)^2 \phi_0 = P_0\phi_0 \tag{26}$$

in which

$$P_0 = \hbar K/4 = D_0/c. \tag{27}$$

A look into (25) reveals that this equation has only one eigenvalue equal to the dissociation energy of the Morse oscillator. In the dissociation state the motion along the time trajectory is not of oscillatory type — its direction is consistent with the arrow of time. Such a situation is familiar for the systems whose temporal evolution is described by the sigmoidal Gompertz function [21]. In the next section two examples of systems of this type will be presented.

## 5 Gompertzian systems

The sigmoidal (S-shaped) functions are widely applied to describe growth of a system, which is retarded and saturated as time continues. They have been widely used to describe the growth of biological systems, for example the growth of organisms, organs, tisses, tumors, bacterial colonies. Among the numerous sigmoidal functions (Avrami, Bertalanffy-Richards, Verhultst, Gyllenberg-Webb, logistic, etc.) describing the growth of biological systems the most popular is the Gompertz function [12]

$$G(t) = G_0 \exp\left[\frac{b}{a}\left(1 - e^{-at}\right)\right]. \tag{28}$$

In the above formulae $a$ is the retardation constant, $b$ denotes the initial growth or the regression rate constant: the sign of $b$ indicates if the system grows (+) or regresses (−). The constant $G_0 = G(t = 0)$ stands for the initial characteristics of the system, for instance, the initial mass, volume, diameter or number of proliferating cells.

It has been demonstrated [21] that function (28) is a solution of the temporal second-order differential equation

$$-\frac{d^2G(t)}{dt^2} + \frac{a^2}{4}\left(1 - \frac{2b}{a}e^{-at}\right)^2 G(t) = \frac{a^2}{4}G(t), \tag{29}$$

which can be given in a dimensionless form

$$-\frac{1}{2}\frac{d^2G(\tau)}{d\tau^2} + \frac{1}{4}\left(1 - e^{-\sqrt{2}\tau}\right)^2 G(\tau) = \frac{1}{4}G(\tau) \tag{30}$$

in which

$$G(\tau) = G_\infty \exp\left[-\frac{1}{2}e^{-\sqrt{2}\tau}\right], \qquad G_\infty = G_0 e^{\frac{b}{a}}, \tag{31}$$

and

$$\tau = (\sqrt{2})^{-1}a(t - t_e), \qquad t_e = -\frac{1}{a}\ln\left(\frac{a}{2b}\right). \tag{32}$$

The sigmoidal functions (Avrami, Fouber et al., etc.) have also been applied to describe the crystallization process. It consists of two stages: (i) nucleation in which molecules come into contact and interact to form ordered structures, and (ii) crystal growth which is the enlargement of the interacting nuclei. Because there are several analogies between crystallization and bacterial growth [23] (creation of bacteria resembles nucleation and crystal's growth, whereas bacterial consumption of nutrients resembles a decrease in supersaturation) the reparametrized Gompertz function

$$f(t) = f_0 \exp\left[-e^{-at+d}\right], \quad a = \frac{\mu e}{f_0}, \quad d = \frac{\mu e\lambda}{f_0} + 1, \tag{33}$$

has been proposed by Zwietering et al. [13] to describe the crystallization process. In the above equation $f_0$ is the maximal value reached, $\mu$ is the maximum specific growth rate defined as the tangent in the inflection point, $\lambda$ denotes the lag time defined as the $t$-axis intercept of that tangent whereas $e = 2.718281...$

The function (33) given in the dimensionless form

$$f(\tau) = f_0 \exp\left[-\frac{1}{2}e^{-\sqrt{2}\tau}\right], \quad \tau = (\sqrt{2})^{-1}a(t - t_e),$$

$$t_e = \frac{1}{a}\left[\ln(2) + d\right], \tag{34}$$

is also a solution of the temporal dimensionless second-order differential equation (30).

A comparison of (30) with (22) reveals that the former has identical form as equation (22) for a space-like quantal Morse oscillator with the anharmonicity constant equal to one, whereas the ground state dissociation solution (23) is identical to within the multiplicative constants $G_\infty = G_0e^{b/a}$ and $f_0$, with the Gompertz (31) and

Zwietering et al. (34) functions. Those results indicate that:

(i) the Gompertzian growth is governed by the equation identical to that of the space-like Feinberg-Horodecki for a time-dependent Morse oscillator with the anharmonicity constant $x_e = 1$,
(ii) the transport of mass in the Gompertzian systems is driven by a time-dependent Morse potential [21],
(iii) the Gompertzian growth takes place in the direction consistent with the arrow of time and resembles dissociation of the anharmonic oscillator,
(iv) the limiting value of the anharmonicity constant $x_e \to 1$ can be used to obtain the micro-macro correspondence relating the quantal space-like coherent states of Morse oscillator with the macroscopic states of the Gompertzian growth.

## 6 Coherent states of the Gompertzian growth

Substituting $x_e = 1$ to (10–12) one gets the space-like annihilation and creation equations which correspond to the equations governing the macroscopic Gompertzian growth and regression [21]

$$\hat{B}^\dagger \hat{B}|v\rangle = 0, \tag{35}$$

$$\hat{B} = \frac{1}{\sqrt{2}}\left[\frac{d}{d\tau} - \frac{1}{\sqrt{2}}e^{-\sqrt{2}\tau}\right],$$
$$\hat{B}^\dagger = \frac{1}{\sqrt{2}}\left[-\frac{d}{d\tau} - \frac{1}{\sqrt{2}}e^{-\sqrt{2}\tau}\right]. \tag{36}$$

The coherent states of the Gompertzian growth are solutions of the annihilation operator [21]

$$\hat{B}|\beta\rangle = \beta|\beta\rangle, \qquad |\beta\rangle = e^{\sqrt{2}\beta\tau}\exp\left[-\frac{1}{2}e^{-\sqrt{2}\tau}\right], \tag{37}$$

whereas the coherent states of the Gompertzian regression are eigenstates of the creation operator $\hat{B}^\dagger$. The states $|\beta\rangle$ minimize the time-energy uncertainty relation [21]

$$(\Delta T)^2(\Delta E)^2 \geq \frac{1}{4}\langle\beta|g(\tau)|\beta\rangle^2,$$
$$[T, \hat{E}] = ig(\tau), \qquad g(\tau) = e^{-\frac{1}{\sqrt{2}}\tau}, \tag{38}$$

yielding

$$(\Delta T)^2(\Delta E)^2 = \frac{1}{4}\langle\beta|g(\tau)|\beta\rangle^2 \tag{39}$$

in which $|\beta\rangle$ is defined by (37), whereas $T = (1/\sqrt{2})e^{-\sqrt{2}\tau}$ represents the temporal Morse coordinate.

Introducing $x_e = 1$ and $\beta = 0$ to (13) and (14) they yield the ground state annihilation equation including the dimensionless Gompertz function of growth

$$\hat{B}|0\rangle = 0 \qquad |0\rangle = \exp\left[-\frac{1}{2}e^{-\sqrt{2}q}\right]. \tag{40}$$

Returning to the original time-coordinate (32), equations (36) can be given in the form of the temporal first-order differential equations ($b > 0$)

$$\frac{dG(t)}{dt} - be^{-at}G(t) = 0,$$
$$G(t) = G_0\exp\left[+\frac{b}{a}\left(1 - e^{-at}\right)\right], \tag{41}$$
$$\frac{dG^\dagger(t)}{dt} + be^{-at}G^\dagger(t) = 0,$$
$$G^\dagger(t) = G_0^\dagger\exp\left[-\frac{b}{a}\left(1 - e^{-at}\right)\right] \tag{42}$$

which represent the growth and regression of the macroscopic Gompertzian systems [22]. They are widely applied in medical and biological sciences, for example to describe a tumor response to chemotherapy [22].

In a similar manner from equations (36) and variable (34) one gets equations describing the crystal growth and regression ($a > 0$)

$$\frac{df(t)}{dt} - ae^{-at}f(t) = 0,$$
$$f(t) = f_0\exp\left[-e^{-at+d}\right], \tag{43}$$
$$\frac{df^\dagger(t)}{dt} + ae^{-at}f^\dagger(t) = 0,$$
$$f^\dagger(t) = f_0^\dagger\exp\left[+e^{-at+d}\right]. \tag{44}$$

The results obtained reveal that the macroscopic equations governing the Gompertzian growth and regression can be derived from the annihilation and creation equations for the space-like Morse oscillator. In particular the Gompertz (31) and Zwietering et al. (34) functions are identical, to within multiplicative constants, to the function (23), representing the ground state solution of the annihilation operator (36) for a space-like Morse oscillator with the anharmonicity constant $x_e = 1$.

## 7 Space-evolution of the Gompertzian coherent states

The space-like Gompertz coherent states (37) satisfy the following relations:

$$|\beta(x)\rangle = |\beta\rangle e^{iPx/\hbar}$$
$$\beta(x) = \frac{1}{\sqrt{2}}\left[T(x) - iE(x)\right]e^{iPx/\hbar}, \tag{45}$$

$$T(x) = \langle\beta(x)|\hat{T}|\beta(x)\rangle \qquad E(x) = \langle\beta(x)|\hat{E}|\beta(x)\rangle, \tag{46}$$

$$T(x) = \sqrt{2}\text{Re}[\beta(x)] = T(0)e^{iPx/\hbar},$$
$$E(x) = \sqrt{2}\text{Im}[\beta(x)] = -E(0)e^{iPx/\hbar}. \tag{47}$$

The states $|\beta(x)\rangle$ assumed to be coherent at the point $x = 0$ remain coherent in all points of space. Such states evolve along localized (classical) time-trajectory being coherent in all points of space. This conclusion becomes clear if we take into account the results obtained in the previous section. For the time-like coherent states of the space-dependent Morse oscillator we have $\Delta Q = \Delta P = const$ [8] in which $Q$ is the spatial variable and $P$ associated momentum. Such states evolve coherently in time being localized on the classical space-trajectory [8]. In the case of the Gompertz coherent states we have $\Delta T = \Delta E = const$. Such states assumed to be coherent at an arbitrary point remain coherent at all points of space. It becomes apparent that the spatial coherence is an immanent feature of the Gompertzian growth. It means that all spatially separated elements of the Gompertzian systems have to be interrelated via long-range interactions permitting the spatially coherent evolution of the system as a whole and evolution of its interconnected subelements [21]. This conclusion is consistent with the Mombach et al. [27] approach in which the Gompertzian model is derivable in the framework of the mean-field theory of cellular growth assuming the presence of long-range (slowly decaying) interactions. In biological systems long-range interactions are mediated through diffusive substances (growth factors), which interact with specific receptors on the surface of the cells.

## 8 Conclusions

The constructed space-like coherent states of the time-dependent Morse oscillator minimize the time-energy uncertainty relation and evolve coherently in space being localized along the classical time-trajectory. Such states are complementary to the ordinary time-like coherent states of a space-dependent Morse oscillator, which minimize the position-momentum uncertainty relation and evolve coherently in time being localized on the classical space trajectory. The anharmonicity parameter $x_e$ can be used to construct the coherent states of the Gompertzian systems, whose growth is described by the Gompertz or Zwietering et al. functions. The growth of the Gompertzian systems is coherent in space: being coherent at an arbitrary point of space, it remains coherent at all points of space. Such a space-like coherence is independent of spatial separation of the subelements of the system, hence the former have to be interrelated via long-range interactions. The space-like character of this connectedness enables the system to form coherent complex patterns in response to external and internal conditions. This response requires self-organization of the system and effective cooperation of all its interconnected subelements.

In 2003 Molski and Konarski [21] showed that the growth of biological systems (organism, organ, tissue, tumor, bacterial colony), characterized by the Gompertz function, is a spatially coherent phenomenon. They derived the temporal second-order differential equation governing the Gompertzian growth, which expressed in dimensionless coordinate had the form identical to that of the quantal Schrödinger equation for the Morse oscillator with anharmonicity constant equal to one. In this work it has been proved that the origin of the Gompertz function is not the time-like Schrödinger equation but the space-like Feinberg-Horodecki one. Only then one may explain why Gompertzian states minimize the time-energy uncertainty relation and not position-momentum one, as it is for time-like solutions of the Schrödinger equation. Having the Feinberg-Horodecki equation introduced, one may also derive equations governing the spatial propagation of the Gompertzian states; in the previous approach it was impossible.

The space-like quantal Feinberg-Horodecki equation for the time-dependent Morse oscillator with the anharmonicity constant equal to one, has identical form as the macroscopic temporal second-order differential equation describing Gompertzian growth. This result is surprising as both equation (29) and its solutions in the form of the Gompertz function (28) are macroscopic formulae and not microscopic ones. We conclude that the Gompertz and Zwietering functions describe the growth of the spatially coherent macroscopic systems in which a large number of subelements collectively cooperate in the ground $v = 0$, $\beta = 0$ coherent mode of growth. This effect resembles the low-temperature Bose-Einstein condensation appearing in the superconductivity and superfluidity phenomena in which a large number of microparticles collectively cooperate sharing the same quantum state [24–26]. In the case of biological Gompertzian systems, the growing cells resemble coupled anharmonic Morse oscillators sharing the same quantum state (mode of growth) in which cells collectively cooperate. It is concluded that the coherent formation of the specific growth patterns in the Gompertzian systems is a result of long-range cooperation between the micro-level (the individual cell) and the macro-level (the system of cells as a whole). The proposed interpretation is consistent with:

(i) the Fröhlich [28] model of macroscopic quantal coherence in biological systems. In this approach a system of coupled oscillators in a heat bath is supplied with energy at a constant rate. When the rate exceeds a certain mean rate then the oscillators condense into one giant dipole whose subelements are spatially interconnected each to other;
(ii) Laird [29] who indicated that the Gompertz function (28) evaluated for the system of proliferating cells can be extrapolated to one cell. It means that it properly describes the growth of the macrosystems composed of a large initial number of proliferating cells ($G_0 = 10^3$–$10^5$) [30] as well as microsystems composed of one cell ($G_0 = 1$) [29];
(iii) Bajzer [31] and Vuk-Pavlović [32] who proved that the Gompertzian growth is a self-similar and allometric process in which the sizes of the growing system at different times are interrelated by a simple power low. Hence, the Gompertz function (28) is a self-similar function, and the selfsimilarity implies invariance of scaling;
(iv) Mombach et al. [27] who derived the Gompertz function of growth in the framework of the cellular mean-field theory, in which the presence of long-range (slowly decaying) interactions is taken into account.

The points (ii) and (iii) indicate that the Gompertz model properly describes the growth of a macrosystem (organism, organ, tumor, bacterial colony, crystal) as a whole and its subsystems (microsystems) composed of the one single cell or molecular aggregate.

The results obtained in this work indicate that the space-like Morse coherent states seem to be a key to understanding of the coherent formation of the specific growth patterns in the Gompertzian systems in particular and the self-organization, cooperation and biological order in general.

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